MODELING THE VOLATILITY OF THE HEATH-JARROW-MORTON MODEL: A MULTI-FACTOR GARCH ANALYSIS †

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Abstract

Based on the nonparametric study of Pearson and Zhou (1999), a parametric HJM model is developed for the forward rate volatility. It allows the volatility of the forward rate with different maturities to react in a different way with the level of forward rate and the forward spread. Specifically, the proposed forward rate volatility function is imbedded into GARCH family models and compared with several widely used HJM volatility specifications. It is shown that the proposed volatility specification performs the best. It is also confirmed that the volatility of forward rate with different maturities depends on the forward rate and the forward spread in a different way.

1. Introduction

Pioneered by Ho and Lee (1986), the arbitrage models have developed into one of the two major frameworks in the term structure literature. They take as given the initial term structure of the interest rate so that the stochastic process of the interest rate is arbitrage free. This method takes advantage of the information of the entire term structure to price contingent claims.

Heath, Jarrow and Morton (HJM) (1992) remarkably generalize the previous studies and develop a multifactor, continuous-time model. They build the model on a given initial forward rate process and use a martingale method to facilitate contingent claims pricing. Given some regularity conditions and under the risk-neutral probability, the contingent claims prices are exclusively determined by the initial term structure and the volatility function of the forward rate. Under this general framework, most of the previous arbitrage models are nested as special cases.

Given the generality of the HJM model, and the critical role the forward rate volatility plays in the model, there have been quite a few studies that attempt to extend and test the HJM model by assuming certain functional forms for the forward rate volatility. The most commonly used specifications are the constant and Square Root models. They are borrowed from the Vasicek (1977) and CIR (1985) models in the interest rate literature and are simple enough to generate closed-form solutions of the options. But studies such as Flesaker (1993) and Amin and Morton (1994) have rejected these models, as they do not adequately reflect the features of the real market data.

More recent studies realize the inadequacy of having a few simple volatility specifications, they propose more sophisticated and more realistic functional forms. Some of these models have been compared and tested by various methods and datasets. Cohen and Heath (1992) compare the performance of several forward rate models by testing their ability to predicting future Treasury security prices. They find that the proportional model performs significantly better than the constant model. Amin and Morton (1994) study the implied interest rate volatilities of six term structure models in the HJM class using Eurodollar futures and options data. It’s shown that while the one-factor models tend to earn larger profits, the two-factor models give closer estimation to the options prices, especially for the long-term options.

Abken and Cohen (1994) conduct a Generalized Method of Moments test on Treasury bond data, which strongly supports the exponentially damped proportional specification which is the combination of the simple exponential model and the linear proportional model. Amin and Ng (1997) study the information content of implied volatility from several volatility specifications of the HJM models. The exponential and linear proportional models outperform the constant, the Square Root and proportional models. Bühler, Uhrig-Homburg, Walter and Weber (1998) compare some forward rate models using German market data. They conclude that the one-factor linear proportional model outperforms the other three models examined, including the two two-factor specifications.

It seems that the conclusion on the performance of a certain volatility specification depends on the method and data used. It's not clear which study yields more convincible results. Pearson and Zhou (1999) make the very first effort to estimate the forward rate volatility function by conducting a nonparametric analysis so that the relationship between the forward rate volatility and the forward rate level as well as the forward spread is developed without being imposed any specification assumption. It provides important guidance for further parametric studies, and also makes it possible to compare and test the parametric models in a general framework.

Using specifications suggested by the results of Pearson and Zhou (1999), this paper develops a parametric model for the forward rate volatility of HJM class by conducting a multi-factor GARCH analysis. In this paper, we construct time-series of the instantaneous forward rates for a range of maturities, investigate the factors that drive the movement of the forward rate volatility and compare the performance of several HJM volatility models in the GARCH family framework. Specifically, we select the GARCH model that best fits the data and imbed in it the five HJM volatility models. The performances of the five HJM models are then examined and compared.

The balance of the paper is organized as follows. Section 2 briefly reviews the HJM framework. Section 3 combines the "level" modeling and GARCH modeling to propose a generalized HJM volatility model in the GARCH framework. Section 4 introduces the econometric approach while Section 5 explains data. Section 6 reports the results of GARCH-family HJM volatility estimation. Section 7 concludes.

2. The HJM Framework

Following Heath, Jarrow and Morton (1992), we consider a trading interval $[0, t]$ for a fixed $t > 0$. Let $f(t,T)$ be the instantaneous forward rate at time t for date $T\,(T > t)$. The period $(T - t)$ is referred to as the time to maturity. The probability space is represented by (Ω, F, P), where Ω is the state space, F is the s -algebra, and P is the probability measure. The instantaneous forward rate $f(t,T)$ is defined by:

$$f(t,T) = -\partial \log P(t,T)/\partial T \qquad \forall T \in [0,t],\ t \in [0,T].$$

where $P(t,T)$ is the time t price of a zero-coupon bond that matures at T. $f(t,T)$ is assumed to follow the Ito process

$$df(t,T) = m(w,t,T)dt + s(w,t,T)dB(t)\,, \tag{1}$$

where $B(t)$ is a Brownian motion, m and s are the "drift" and "volatility" (or diffusion) functions, respectively, and $w \in \Omega$ indicates the possible dependence of the drift and volatility functions on the entire history of the process.

The evolution of the forward rate of all maturities is simultaneously and exogenously determined. Once the forward rate term structure is derived, the dynamics of the bond price can be obtained as the following (see HJM(1992)):

$$dP(t,T) = [f(t,t) + b(t,T)]P(t,T)dt + a(t,T)P(t,T)dB(t) \tag{2}$$

where

$$a(w,t,T) = -\int_t^T s(w,t,u)du$$

$$b(w,t,T) = -\int_t^T m(w,t,u)du + (1/2)\,a(w,t,T)$$

Given the initial term structure of the forward rate and some regularity conditions, there exists a unique equivalent martingale probability measure or risk-neutral probability Q, which implies that the drift term can be expressed by the volatility functions:

$$m^Q(w,t,T) = s(w,t,T)\int_t^T s(w,t,v)dv\,.$$

Thus, under Q, the forward rate $f(t,T)$ evolves according to the process

$$df(t,T) = m^Q(w,t,T)dt + s(w,t,T)d\widetilde{B}(t),\,. \tag{3}$$

where $\widetilde{B}(t)$ is the Brownian motion under the risk-neutral probability. It's the difference between the original Brownian motion and the sum of the market price of risk over time, i.e.,

$$\widetilde{B}(t) = B(t) - \int_0^t g(v)dv, \tag{4}$$

where $g(t)$ is the market price for risk at time *t*. Now what determine the entire term structure of the forward rates are the initial term structure of the forward rate and the volatility function of the forward rate.

In modeling forward rate volatility, there are in general two approaches. One is to assume that the volatility changes with the forward rate level, which we call the "level" method, the other is to assume that the volatility evolves from its own history, which we call the GARCH method.

3. Volatility Modeling

3.1. The GARCH Models

GARCH models are used as a successful treatment to the financial data which often demonstrate time-persistence, volatility clustering and deviation from the normal distribution. Among the earliest models is Engel (1982) linear ARCH model, which captures the time varying feature of the conditional variance. Bollerslev (1986) develops Generalized ARCH (GARCH) model, allowing for persistency of the conditional variance and more efficient testing. Engle and Bollerslev (1986) invent the Integrated GARCH (IGARCH) model that provides consistent estimation under the unit root condition. Engle, Lilien, and Robins (1987) design the ARCH-in-Mean (ARCH-M) model to allow for time varying conditional mean. Nelson's (1990b) Exponential GARCH (EGARCH) model allows asymmetric effects and negative coefficients in the conditional variance function. The leveraged GARCH (LGARCH) model documented in Glosten, Jagannathan and Runkle (1993) takes into account the asymmetric effects of shocks from different directions.

As far as the HJM volatility modeling is concerned, we are interested in finding some continuous-time models that best describe the dynamics of the time series. Being discrete-time by nature, how well can the GARCH class models approximate the continuous-time process suggested in the previous section? Nelson (1990b) shows that the GARCH (1,1) model and

EGARCH model converge to continuous time Ito diffusion processes, as the sample interval goes to zero. Nelson (1992), Nelson and Foster (1994) also show that ARCH models fitted to high frequency data provide optimal and consistent estimates of true volatility underlying a given observation system. Fornari and Mele (1995) provide a summary of ARCH models and their diffusion limits. Duan (1995) shows that under some preference and distribution assumptions, the GARCH model converges to its diffusion limit and can explain some pricing error in the Black-Scholes option model. It seems that as long as the model and the data frequency are properly selected, the ARCH models approximate the true processes reasonably well.

3.2. The Level Models

The level models have been widely proposed and used in the interest rate term structure literature, such as the Constant model (Ho and Lee (1986)), the Square Root model (Cox, Ingersoll and Ross (1985)) and the Exponential model (Vasicek (1977)).

In the HJM framework, the arbitrage-free assumption requires that the volatility function of the forward rate be bounded (see HJM (1992) page 80, C1 (iii)). The Constant model and the exponentially damped model are both bounded. HJM (1992) propose a constraint proportional function of the forward rate, which allows the volatility to change linearly with the forward rate when the forward rate is low, but is capped by a constant when the forward rate is high. The Square Root model can be constrained in a similar manner. Thus the four models that have been widely used in the existing literature become the benchmarks of our study and are thus included. Their detailed functional forms are provided in Table 1.

The nonparametric analysis of Pearson and Zhou (1999) document that the volatility functions of the forward rate adopt different forms for different maturities. They change with the forward rate level and the forward rate spread. Specifically, the volatility increases monotonically with the forward rate at a faster speed for short maturities, but becomes a combination of the convex and concave function for medium and long maturities, i.e., it increases with the forward rate and then decreases with it at moderate forward rate levels. With respect to the forward rate spread, the volatility increases with it in general, but for long maturities, it decreases at the moderate spread levels.

Given the features of the forward rate volatility discovered in Pearson and Zhou (1999), we propose the following parametric model:

$$\sigma(\cdot) = \sigma_0 f(t,T)^{\sigma_1} \left| f(t,T) - c \right|^{\sigma_2} e^{\lambda(T-t)} \tag{5}$$

For all coefficients, the sign and size depend on time to maturity. For simplicity, we omit this dependency in equation (5). This specific funtional form is adopted due to the following considerations: first, we want to garantee the boundary condition that the volatility at zero forward rate equals zero, which avoids deriving negative forward rates with positive possibilities. Second, the time to maturity $(T-t)$ plays a role in the movement of the forward rates, depending on the sign of λ, which we expect to be negative. Third, the term $f(t,T)^{\sigma_1}$ is included to allow for a power functional form. This is posible when the coefficient σ_1 is positive. To avoid infinitive volatility at zero forward rate, we define σ_1 to be strictly positive. Fourth, the term $\left| f(t,T) - c \right|^{\sigma_2}$ is included to allow for a combination of the concavity and convexity with respect to the forward rate. This is possible when c is positive and within the range of the forward rate levels. When c is negative, this term will only show convexity or concavity, depending on the sign of σ_2. When c is positive but bigger than the maximum forward rate level, we need to guarantee the term $f(t,T) - c$ be raised by a non-integer power, such as 0.5, so we take its absolute value and let it be raised to power σ_2. To qualify for the HJM volatility function, the model specified in Equation (5) should also be constrained so that the volatility doesn't explode in finite time. The volatility specification in Equation (5) is referred to as the Combination model. Its constrained functional form is provided in Table 1.

The four alternative HJM volatility models (or the corresponding constrained forms) summarized in Table 1 are in fact the special cases of the Combination model. Specifically, when σ_1, σ_2 and λ are zero, the function collapses to the Constant model. When σ_1 and σ_2 are zero, it becomes the Exponential model. When σ_2 and λ are zero, it can be either the Proportional model or the Square Root model, depending on the parameter σ_1 . Besides, it nests most of the specifications that previous studies have proposed and examined. A list of the existing specifications are provided in Table 2.

As suggested in Pearson and Zhou (1999), the forward rate spread is another factor that affacts the forward rate volatility. The forward rate spread is defined as the difference between a forward rate with long maturity and a forward rate with short maturity. It's regarded as a measure of the slope of the forward yield curve. It represents a term premium, a compensation for the risk of holding a longer-term contract. According to the results in Pearson and Zhou (1999), the spread effect varies with maturities. Specifically the volatility is in general an increasing function of the forward spread for short-maturity series, but becomes a combination of convex and concave function for long-maturity series. As an extension to the parametric model described in (5), we include the forward rate spread as the third factor and examine its role in the volatility function:

$$s(\cdot) = g\left(f(t,T), T-t, f(t,T)^l - f(t,T)^s\right) \tag{6}$$

where $f(t,T)^l$ is a forward rate with long maturity and $f(t,T)^s$ is a forward rate with short maturity.

3.3. GARCH Imbedded Level Models

As both the level models and the GARCH models have their own merits in treating the volatilities, it is difficult to pick one model without comparing their contributions. To do this, we combine the two models into a more generalized framework and examine its power in capturing the forward rate volatility movement. Thus the combined level and GARCH (*p*, *q*) model will look like the following:

$$f_t = a_0 + a_i f_{t-1} + e_t$$

$$e_t = h_t \sqrt{h_t}$$

$$h_t = c_0 + \sum_{i=1}^{p} p_i h_{t-i} + \sum_{i=1}^{q} q_i e_{t-i}^2 + \text{Level Model}, \tag{8}$$

where f_t is the instantaneous forward rate at time t with the time to maturity of $(T\text{-}t)$, h_t is the conditional variance of the forward rate at time t, and h_t is assumed to follow the standard normal distribution. Besides the GARCH model, we also experiment with various extensions of the GARCH family models, such as EGARCH and LGARCH[1]. As mentioned before, the EGARCH model allows for asymmetric effects and negative coefficients in the conditional variance function, which greatly extends the analyzing power and adds to the flexibility. The LGARCH model allows for different effects of shocks from different directions. We will choose the GARCH-family model that best fits the data.

[1] IGARCH is not selected as it is used in the unit-root case, when the volatility grows without bound. But this

3.3.1. The Level and Time-to-Maturity Effect

We first examine the effects of the forward rate level and time-to-maturity (maturity effect hereafter) on the conditional volatility of the forward rate. Specifically, we imbed the HJM volatility functions described in Section 2 into the GARCH family conditional variance functions. For example, the GARCH (1, 1) model that imbeds the Square Root model has the following conditional variance function:

$$h_t = c_0 + p h_{t-1} + q e_{t-1}^2 + \left(s_0 f_{t-1}^{1/2}\right)^2. \tag{9}$$

The square term is due to the fact that the square root specification is for the "standard deviation" of the forward rate and h_t is the variance of the forward rate. Similarly, the EGARCH (1, 1) model that imbeds the Exponential model has the following conditional variance function:

$$h_t = \exp(c + p\log(h_{t-1}) + q g_{t-1}) + \left[s_0 e^{l(T-t)}\right]^2,$$

$$g_{t-1} = \left| e_{t-1} \big/ \sqrt{h_{t-1}} \right| - \sqrt{2/p} - l e_{t-1} \big/ \sqrt{h_{t-1}}\,. \tag{10}$$

And the LGARCH (1, 1) model that imbeds the Combination model is:

$$h_t = c_0 + p h_{t-1} + q e_{t-1}^2 + l I_{t-1} e_{t-1}^2 + \left[s_0 f_{t-1}^{s_1} (f_{t-1} - c)^{s_2} e^{l(T-t)}\right]^2, \tag{11}$$

where I_{t-1} equals 1 when $e_{t-1} < 0,$ and 0 otherwise. The variance function of the Constant model coincides with its GARCH-family counterpart.

3.3.2. The Forward Rate Spread Effect

We next introduce the forward rate spread into the conditional volatility functions as the third factor and examine its contribution. Specifically, the GARCH (1, 1) model that corresponds to the Combination model is[2]:

$$h_t = c_0 + p h_{t-1} + q e_{t-1}^2 + \left[s_0 f_{t-1}^{s_1} \left| f_{t-1} - c \right|^{s_2} e^{l(T-t)} + s_0 (f_{t-1}^{l} - f_{t-1}^{s})\right]^2 \tag{12}$$

where f_{t-1}^{l} is the forward rate with a long maturity at time *t-1*, and f_{t-1}^{s} is the forward rate with a short maturity at time *t-1*.

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violates the condition of the HJM volatility.

[2] We've tried several non-linear specifications of the forward spread. But they can't generate convergence.

4. The Econometric Approach

4.1. Maximum Likelihood Method

As pointed out by Bera and Higgins (1993), the GARCH models are most often estimated by maximum likelihood method. We thus adopt it in this study as well. The log likelihood function of the GARCH model based on previous period's information $f_t|\Psi_{t-1} \sim N(a_0 + a_1 f_{t-1}, h_t)$ is given by

$$l(q) = \frac{1}{T}\sum_{t=1}^{T} l_t(q),$$

where $q = (x', g)'$, with x and g the conditional mean and conditional variance parameters respectively, and

$$l_t(q) = const. - \frac{1}{2}\log(h_t) - \frac{e_t^2}{2h_t^2}.$$

The likelihood function provided above is maximized using Berndt, Hall, Hall and Hausman (1974) (BHHH) numerical algorithm.

4.2. Tests

Specification test is conducted by checking the normality of the standardized residual:

$$e_t^* = e_t \big/ \sqrt{h_t}\ .$$

If the conditional variance function is well specified, then the standardized residual should be close to the white noise. Jarque-Bera test is used to detect the deviation from normality. Ljung-Box Q-test is used to detect the serial correlation in the standardized residual and the squared standardized residual. F-test is used to detect ramaining ARCH effect.

Engle and Ng (1993) develop the sign bias test to detect the asymmetric impact of the lagged negative and positive shocks on the conditional variance. They also design the negative size bias test and positive size bias test to determine whether shocks from different directions and with different magnitudes have different impacts on the conditional variance. These tests are also used in this paper.

Likelihood ratio test is used to examine the relative performance of each of the four HJM models with respect to the Combination model.

Lastly, we test the equality of the coefficient estimates between each pair of series. Since the residuals of different series do not have equal or known variances in general, we use the modified the Chow-type test proposed by Toyoda (1974), which only requires larger sample size for at least one series.

5. The Data

We estimate the volatility functions using the estimates of the "instantaneous" forward rates for a range of maturities. These estimates of the "instantaneous" forward rates are constructed from the daily settlement prices of the Eurodollar futures contracts (based on 3-month LIBOR) and 1-month LIBOR futures contracts traded on the Chicago Mercantile Exchange. The Eurodollar futures contracts started trading in December 1981, when only 3 and 6-month contracts were available. Starting in 1990, the maturities of the Eurodollar contracts have extended out to at least 4 years, and currently extend out to 10 years. The 1-month Eurodollar futures contracts are also available from 1990.

Some advantages of using Eurodollar futures data are discussed in Jegadeesh and Pennacchi (1996). Eurodollar and 1-month LIBOR futures contracts are very actively traded with a very small bid-ask spread. Trading stops in all contracts at the same instant, at which time final settlement prices for all contracts are determined essentially simultaneously, eliminating concerns about the possible non-synchroneity of prices. In addition, because 3-month LIBOR is a common index for floating rate instruments such as interest rate swaps and floating rate notes, the Eurodollar contracts are widely used for hedging and arbitrage, linking the Eurodollar term structure to the term structure of swap rates. A further advantage pointed out by Amin and Morton (1994) is that Eurodollar and 1-month LIBOR futures contracts are cash-settled, which avoids some delivery and timing problems that are inherent in the Treasury bond and note futures contracts.

We construct the daily instantaneous forward rates with maturities from 3-month up through 48-month. The details of the data construction are provided in the Appendix. Figure 1 shows the time series of the 3-month, 12-month, 24-month, and 48-month instantaneous forward rates, while Figure 2 shows plots of daily rate changes. Some summary statistics of the selected series of forward rates are provided in Table 3. We observe that the sample mean increases with maturity, all series are slightly skewed to the right and have slightly thinner tails. The up to six-lagged autocorrelation coefficients of each series are all greater than 0.96, which shows a strong time persistency. But the autocorrelation coefficients of the rate changes are small and change signs from time to time.

6. The Empirical Results

6.1. GARCH-family Model Selection

We start with the generic GARCH family models[3] to examine whether these models fit the forward rate data well. Specifically, we estimate the AR (1)-GARCH (1,1) family models for all 16 series of forward rates by using the maximum likelihood method and BHHH algorithm, without inserting the HJM volatility specifications. The results of the 48-month series are provided in Table 4, which shows the pattern that most other data series possess. As we can see, in general all three models do a good job in capturing the ARCH effect. Though there is still some serial correlation left in the residual, the serial correlation in the squared residual and the size and sign bias are very well rectified. Among the three models, the GARCH model has the lowest log likelihood functional value. It also generates a larger skewness and kurtosis than the EGARCH model. The LGARCH model also has lower log likelihood function value and higher skewness and kurtosis than EGARCH. Besides, the leverage effect in the LGARCH model is not significant[4]. As this is the case for most other data series, it is clear that the LGARCH model does not bring extra treatment to the data than the general GARCH model. Overall, we believe that the EGARCH model fits the data the best so we will use it in the following analysis[5].

6.2. The Level and Maturity Effects

We imbed the five HJM models in the EGARCH model and compare their performance. The results are summarized in Tables 5 to 8.

[3] For the generalized AR (s)-GARCH (p, q) model, we've tried several combinations of *s*, *p* and *q* and found that the AR(1)-GARCH (1, 1) family models provide good approximation in most cases. This is consistent with the results in some related studies such as Milhφj (1990).
[4] The computation is done with the software Rats. For the BHHH method, the *p*-values are computed using the product of the first derivatives. As shown in related studies, it is consistent for the matrix being estimated, and it converges to the "true" value as sample size grows. Given the relatively large number of observations, this should not be a concern.
[5] We have also imbedded the Combination model (equation (5)) into the three GARCH models to compare their

Table 5 reports the EGARCH estimation results for the 6-month forward rate from April 1990 to October 1998. All the five HJM models have successfully captured the serial correlation as well as sign and size bias in the conditional variance function. The Square Root model has the lowest log likelihood function value. The Square Root coefficient s_0 is negative and is significantly different from zero but is small in magnitude. The model is rejected by the likelihood ratio test with respect to the Combination model, showing that the Square Root specification does not provide good description to the volatility of the short-term forward rate. The Constant model and Proportional model have higher log likelihood function values than the Square Root model. In the Proportional model, the proportional coefficient s_0 is small but insignificant, after counting for the lagged conditional variance effect p and the shock effect q. Both models are rejected by the likelihood ratio test. Among the four alternative HJM volatility models, the Exponential model has the highest log likelihood function value, with s_0 positive and significant, and a large negative and significant l, indicating a decreasing relationship between the volatility and the maturity of the forward rate. Along with the other three models, the Exponential is also rejected by the likelihood ratio test. The Combination model has the highest log likelihood value. Both power coefficient estimates s_1 and s_2 are positive, indicating a positive relationship between the forward rate volatility and the forward rate level. However, after counting for the lagged conditional variance effect and the shock effect, there is only one coefficient that is significant – the maturity effect l. The maturity effect is negative, but is much smaller than that in the Exponential model. As the rest of the coeffient estimates are insignificant, it seems that the volatility of 6-month forward rate is not sensitive to the changes in the forward rate levels.

Table 6 reports the EGARCH estimation results for the 12-month forward rate from April 1990 to October 1998. In the Constant model, all the estimates are significant. The Square Root model and the Proportional model have very similar functional values to the Constant model, with the coefficient estimates s_0 small and insignificant, after counting for the lagged conditional variance effect and the shock effect. The Exponential model and the Combination model have the highest function values. Both have positive and significant s_0 estimates, and negative and significant l estimates, indicating a decaying maturity effect on the forward rate

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performance for the data. Again, the EGARCH model does the superior job.

volatility. Despite the significant p and q effects, all the coefficient estimates in the Combination model are significant, with positive estimates for both s_1 and s_2, confirming the positive relationship between the volatility and the forward rate levels. Given that the estimates for s_1 and s_2 are both larger than one, the volatility becomes a convex funtion of the forward rate level. Of the four alternative HJM models, only the Exponential model fails to be rejected with respect to the Combination model. It shows that other than the Combination model, the Exponential model best captures the movement of the the 12-month forward rate.

Table 7 reports the EGARCH estimation results for the 24-month forward rate from April 1990 to October 1998. As is the case in Table 6, the Constant, Square Root and Proportional models have the lowest log likelihood function values. With most of the estimates insignificant, they are all rejected by the likelihood ratio test. Besides, the Constant and Proportional models do not capture the serial correlation in either the residuals or the squared residuals. The Exponential model has higher log likelihood functional value than the other alternative HJM models, with a small and negative maturity effect. As is the case for the 12-month forward rate, it is the only model that is not rejected by the likelihood ratio test with respect to the Combination model. The Combination model again has the highest log likelihood functional value, with all the coefficients small but significant. With positive estimates for s_1 and s_2 and negative estimate for c, it shows a monotonically increasing relationship between the volatility and the forward rate. As both the estimates of s_1 and s_2 are much smaller than one, the volatility is clearly a concave funtion of the forward rate. There is again a negative and dominant maturity effect.

Table 8 reports the EGARCH estimation results for the 48-month forward rate from April 1990 to October 1998. The Square Root, Proportional and Exponential models are rejected by the likelihood ratio test. The Exponential model has significantly lower functional value than any other model and becomes the worst specification for the volatility of long-term forward rates. The Constant model is the only alternative model that is not rejected by the likelihood ratio test, with very close functional value to that of the Combination model. The Combination has all conditional variance coefficients significant, with positive estimates for s_1 and s_2 and negative estimate for c, showing that the volatility of the long-term forward rate is a concave function of the level of the forward rate. However, the estimate of s_0 is negative though small, which generates a decreasing relationship between the volatility and the level of forward rate.

Compare the results of Combination model in Tables 5-8, we observe obviously different patterns for different data series.[6] For shorter maturities, the forward rate volatility tends to be independent to the change of forward rate level. For medium maturities, the forward rate volatility becomes a convex function of the forward rate level so the Exponential model is a good approximation to the forward rate volatility. For long maturities, the volatility tends to become a concave function of the forward rate level, it even tends to be negative when the maturity becomes very long. As the magnitude of the forward rate volatility is smaller for long-maturity data, the Constant model becomes a good approximation to the forward rate volatility. For all data series, the volatility is a decreasing function of the time to maturity.

6.3. The Spread Effect

We define the forward rate spread as the difference between the 48-month forward rate and the 6-month forward rate. We first isolate the forward rate spread effect by assuming that in equation (12), all the other coefficients are zeros except s_0, so the HJM volatility function looks like the following:

$$s(\cdot) = s_0\left(f(t,T)^l - f(t,T)^s\right). \tag{13}$$

The results are reported in Table 9. For all of the four series, the spread effect is small, positive and significant. The coefficient estimate of s_0 is the highest for the 12-month rate and becomes lower for longer maturity data series.

We next combine the level effect, the maturity effect and the spread effect and estimate them in one model as specified in equation (12) but assuming s_0 is zero[7]. The results are reported in Table 10. For the 6-month forward rate, after counting for small and negative spread effect and maturity effect, the level effect becomes insignificant. For the 12-month forward rate, after counting for small and negative spread effect and maturity effect, the level effect remains significant. Specifically, the volatility becomes a concave function of the forward rate level. For the 24-month forward rate, after counting for a small and positive spread effect and a small and

[6] To confirm this, we conduct a Chow-type test which allows unequal and/or unknown variances to compare the coefficient estimates for different series. The results show that they are significantly different. For each pair of the three series studied, the F-statistics are significantly larger than the critical value, strongly reject the hypothesis that the coefficients of different time series are the same. This indicates that the volatility of different forward rate series reacts in a different way to the changes of its driving forces.

[7] We have tried to estimate the Equation (12) without taking off the first term of the level effect. But no convergence

negative maturity effect, the level effect remains significant. Specifically, the volatility becomes a decreasing function of the forward rate level. For the 48-month forward rate, after counting for small and negative spread effect and maturity effect, the level effect remains significant. Specifically, the volatility becomes a decreasing function of the forward rate level.

7. Conclusion

This paper proposes a more general and realistic parametric model for the forward rate volatility, based on the nonparametric results in Pearson and Zhou (1999). It is confirmed that the volatility changes with the forward rate level differently for forward rates with different maturities. Specifically, before counting for the spread effect, the volatility is a convex function of the forward rate level for medium-maturity data series, but is a concave function for both the short-maturity and long-maturity data series. For all four data series, the volatility decreases with the time-to-maturity. After adjusting for the level and maturity effects, the forward rate spread effect becomes small and changes signs cross maturities.

The contribution of this study is to justify a more general and realistic volatility model for the HJM framework. It improves the understanding on the driving forces of the forward rate volatility and their influences on the forward rate volatility across maturities. On the basis of this new HJM forward rate volatility model, the options prices can be derived more accurately.

A natural extension of this study is to conduct a multivariate GARCH analysis, examining the volatilities of all maturities simultaneously with respect to the forward rate spread as well as time-to-maturity. This makes sense as they all respond to the same set of factors and their covariances are not zeros in general.

can be achieved.

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Appendix: Instantaneous Forward Rate Construction

We first need to understand the relationship between the futures rate and the corresponding forward rate. In the context of specific models, some previous studies (e.g., Grinblatt and Jegadeesh (1996)) determine the difference between the interest rate implied from the futures contracts and the actual implied forward rate, commonly known as the "convexity bias." This bias is due primarily to the fact that the futures contracts settle gains and losses daily, while forward contracts are settled only at maturity. This, together with the asymmetric effect of the interest rate changes on bond prices, results in a gap between the interest rate implied from the futures contracts and the "true" implied forward rates. The former is usually a few basis points higher than the latter, and the difference increases with maturity. Burghardt and Hoskins (1995) document this relationship, and suggest an approximate procedure to adjust the implied futures interest rates that does not depend on any specific model. In constructing our estimates of instantaneous forward rates, we obtain the forward rates using the procedure of Burghardt and Hoskins (1995).

To construct the instantaneous forward rates, we start with the daily prices of the Eurodollar and 1-month LIBOR contracts from April 1990 to October 1998, convert the futures prices into (continuously compounded) yields based on the following,

$$y(T) = 1 - F(T,T)/100,$$

where $F(T,T)$ is the price of futures that maturies at time T, $y(T)$ is the three-month yield on Eurodollar time deposits. We then convert the yields into the corresponding forward rates based on the well known "convexity bias" (see, e.g., Burghardt and Hoskins (1995)). We then "chain" together the forward rates in order to build the term structure. Specifically, for day t, let $t+s_1$, $t+s_2$, …, $t+s_\ell$ denote the last trading dates of the 1-month LIBOR contracts, and let $t+t_1$, $t+t_2$, …, $t+t_m$ denote the last trading dates of the Eurodollar contracts.[8] Starting from the last trading date $t+s_1$ of the first 1-month LIBOR contract, we construct the forward rates

[8] The final settlement value of the Eurodollar and 1-month LIBOR contracts is based on either 3 or 1-month LIBOR quoted on the last trading date, for a deposit period beginning two business days after the last trading date. Thus, the forward rates we construct are actually for the times $t + t_1 + 2$ business days, $t + t_2 + 2$ business days, etc. The discussion in the text ignores this settlement convention in order to prevent the description from becoming needlessly complicated. However, the algorithms used to construct the forward rates incorporated the settlement conventions of the interbank market.

$f(t, t+s_1, t+s_2)$, $f(t, t+s_1, t+s_3)$ …, $f(t, t+s_1, t+t_1)$, where we stop using the 1-month LIBOR contracts at $t+t_1$, the last trading date of the first Eurodollar contract.[9][10] From that date, we use the Eurodollar contracts to construct the forward rates $f(t, t+s_1, t+t_2)$, $f(t, t+s_1, t+t_3)$, …, $f(t, t+s_1, t+t_m)$. The result of this process is a set of forward rates from $t+s_1$ to $t+s_2$, …, $t+t_1$, $t+t_2$, …, $t+t_m$. To these forward rates, we fit a cubic spline[11] to obtain the entire term structure from $t+s_1$ to $t+t_m$. Finally, we differentiated the spline function at the points that correspond to actual time to maturity for each Eurodollar contract. At the same time, we keep track of the maturity of the Eurodollar contract each forward rate is converted from and sort the forward rates based on the maturity of the Eurodollar contracts. This gives us 16 series of daily instantaneous forward rates with maturity of 3 through 48 months, with actual time to maturity, covering the time period from April 1990 to October 1998.[12]

[9] This involves using at most three of the 1-month LIBOR futures contracts.

[10] Because both contracts stop trading two business days before the third Wednesday of the month, the last trading date of the first Eurodollar contract always coincides with the last trading date of one of the 1-month LIBOR contracts. One issue is that the maturity date of the deposit underlying a contract often does not coincide exactly with the last trading date of the next contract. In constructing the forward rates we assumed that it does. This has virtually no impact on the term structures we construct.

[11] We use the "natural" boundary condition that the second derivative of the spline function be zero at the endpoints.

[12] The shortest time to maturity is at least 30 days from the first I-month LIBOR contract so that the forward rates we constructed would not be affected by the spline boundary condition.

Table 1
HJM Volatility Models of the Forward Rate
And the Corresponding GARCH Conditional Variance Functions

Panel A: HJM Forward Rate Volatility Models

Model	Functional Form
Constant	s_0
Square Root	$s_0 \min\left(f(t,T)^{1/2}, M\right)$
Proportional	$s_0 \min\left(f(t,T), M\right)$
Exponential	$s_0 \min\left(e^{l(T-t)}, M\right)$
Combination	$s_0 \min\left(f(t,T)^{s_1} \lvert f(t,T)-c \rvert^{s_2} e^{l(T-t)}, M\right)$

Panel B: The EGARCH conditional variance functions that imbed the HJM models

Model	Functional Form
Constant	$h_t = \exp(c + p\log(h_{t-1}) + qg_{t-1})$
Square Root	$h_t = \exp(c + p\log(h_{t-1}) + qg_{t-1}) + \left[s_0 f(t,T)^{1/2}\right]^2$
Proportional	$h_t = \exp(c + p\log(h_{t-1}) + qg_{t-1}) + \left[s_0 f(t,T)\right]^2$
Exponential	$h_t = \exp(c + p\log(h_{t-1}) + qg_{t-1}) + \left[s_0 e^{l(T-t)}\right]^2$
Combination	$h_t = \exp\left(c_0 + p\log(h_{t-1}) + qg_{t-1}\right) + \left[s_0 f_{t-1}^{s_1} \lvert f_{t-1} - c \rvert^{s_2} e^{l(T-t)}\right]^2$

Panel A summarizes the five HJM forward Rate volatility models. Some of them are constrained by a large constant so as to fulfill the HJM regularity condition.
Panel B reports the EGARCH conditional variance functions that imbed the HJM volatility models listed in Panel A. For all functions,

$$g_{t-1} = \left| e_{t-1} / \sqrt{h_{t-1}} \right| - \sqrt{2/p} - l e_{t-1} / \sqrt{h_{t-1}}$$

Table 2
Forward rate volatility models in the existing literature

Specification	Reference
s_0	Cohen and Heath (1992) Flesaker (1993) Amin and Morton (1994) Amin and Ng (1997) Bühler, Uhrig-Homburg, Walter and Weber (1998)
$s_0 f^{1/2}$	Amin and Morton (1994) Amin and Ng (1997)
$s_0 f$	Amin and Morton (1994) Amin and Ng (1997)
$s_0 + s_1(T-t)$	Amin and Morton (1994)
$\left[s_0 + s_1(T-t)\right]f$	Amin and Morton (1994) Amin and Ng (1997) Bühler, Uhrig-Homburg, Walter and Weber (1998)
$s_0 \exp\left[-\ l(T-t)\right]$	Amin and Morton (1994) Amin and Ng (1997)
$s_0 \min\left[f, M\right]$	Cohen and Heath (1992) Abken and Cohen (1994)
$s_0 g(T-t) \min\left[f, M\right]$	Cohen and Heath (1992) Abken and Cohen (1994)
$s_{0i} g_i(T-t) \min\left[f, M\right]$	Cohen and Heath (1992)
$s_0 \exp\left[-k(T-t)\right] r^g$	Bliss and Ritchken (1995)

Table 3
Summary Statistics

A. Some summary statistics for the forward rates

Maturity	Mean	Standard Deviation	Skewness	Kurtosis	AR1	AR3	AR6
3 month	0.053	0.0134	0.23	2.90	0.996	0.989	0.978
6 month	0.056	0.0130	0.17	2.75	0.996	0.987	0.974
12 month	0.064	0.0131	0.26	2.68	0.994	0.979	0.960
18 month	0.072	0.0136	0.40	2.56	0.994	0.982	0.965
24 month	0.076	0.0135	0.44	2.44	0.996	0.987	0.974
30 month	0.078	0.0139	0.49	2.35	0.995	0.984	0.967
36 month	0.079	0.0142	0.46	2.25	0.995	0.985	0.972
42 month	0.081	0.0146	0.44	2.15	0.994	0.982	0.964
48 month	0.082	0.0139	0.37	2.08	0.993	0.982	0.966

B. Some summary statistics for the forward rate changes

Maturity	Mean	Standard Deviation	Skewness	Kurtosis	AR1	AR3	AR6
3 month	-.00002	.00072	1.58	98.92	.050	.015	-.021
6 month	-.00002	.00070	2.30	34.23	.030	.001	-.022
12 month	-.00002	.00081	.76	10.25	.068	-.012	-.045
18 month	-.00002	.00080	0.80	9.26	.109	-.005	-.046
24 month	-.00002	.00074	0.70	8.16	.118	-.019	-.053
30 month	-.00002	.00067	0.50	6.57	.123	-.016	-.057
36 month	-.00002	.00063	0.50	6.25	.124	-.026	-.063
42 month	-.00002	.00062	0.45	5.90	.124	-.010	-.058
48 month	-.00002	.00061	0.47	5.76	.119	-.015	-.055

Means, standard deviations, skewness and kurtosis of the daily Eurodollar instantaneous forward rates and their daily changes with selected maturities are computed. Up to six-lagged autocorrelation coefficients of selected forward rate series are also provided. The forward rates are from April 1990 to October 1998. There are 2143 observations in each series.

Table 4
Preliminary Model Selection

	GARCH (1,1)	EGARCH (1,1)	LGARCH (1,1)
c_0	.00 (.00)	-.27 (.00)	.00 (.00)
q	.04 (.00)	.10 (.00)	.03 (.00)
p	.95 (.00)	.98 (.00)	.95 (.00)
l	16.09 (.62)	.08 (.18)	.01 (.11)
skew	.34	.31	.34
kurt.	5.36	5.16	5.33
log*L*	12866.6	12871.4	12867.3
$Q(12)_e$	61.7 (.00)	61.3 (.00)	61.2 (.00)
$Q(12)_{e^2}$	10.5 (.48)	11.3 (.42)	9.8 (.55)
Bias Test	.4 (.74)	.3 (.85)	.5 (.66)

Results are for the 48-month forward rate series from January 1990 to October 1998, which represent those for the other series. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t \sqrt{h_t}, \;\; h_t \sim N(0,1).$$

The conditional variance equation in the GARCH model is

$$h_t = c + ph_{t-1} + qe_{t-1}^2$$

The conditional variance equation in the EGARCH model is

$$h_t = \exp(c + p\log(h_{t-1}) + qg_{t-1}),$$

$$g_{t-1} = \left| e_{t-1} / \sqrt{h_{t-1}} \right| - \sqrt{2/p} - le_{t-1} / \sqrt{h_{t-1}}$$

The conditional variance equation in the LGARCH model is

$$h_t = c + ph_{t-1} + qe_{t-1}^2 + lI_{t-1}e_{t-1}^2.$$

The *skew* and *kurt* are the skewness and kurtosis of the residuals. The $Q(12)_e$ is the Ljung-Box Q-Test for the serial corraltion in the residuals, $Q(12)_{e^2}$ is the Ljung-Box Q-Test for the serial corraltion in the squared residuals, and the Bias Test is the joint test for sign and size bias. The numbers in the parenthesis are the *p*-values.

Table 5
HJM Model Comparison: 6-month Rate

	Constant	Square Root	Proportional	Exponential	Combination
c_0	-10.55 (.00)	-65.85 (.83)	-0.07 (.00)	-.04 (.00)	-0.03 (.91)
q	-.01 (.62)	.0003 (1.00)	.02 (.00)	.03 (.00)	.58 (.00)
p	.27 (.00)	-3.21 (.88)	.99 (.00)	1.00 (.00)	1.00 (.00)
l	16.09 (.62)	-134.33 (1.00)	-.91 (.00)	-0.49 (.00)	.01 (.64)
s_0		-.0003 (.00)	.00 (1.00)	1.01 (.00)	.0005 (1.00)
s_1					.11 (.88)
s_2					-.03 (1.00)
c					-1.09 (1.00)
l				-47.48 (.00)	-.02 (.00)
log$L$	12523.2	12452.9	12532.5	12615.2	12666.0
Likelihood Ratio Test	285.6 >11.1	426.2 >9.49	267.0 >9.49	101.6 >7.81	
$Q(12)_e$	11.1 (.44)	8.6 (.65)	8.2 (.69)	7.4 (.76)	14.9 (.19)
$Q(12)_{e^2}$	1.1 (1.00)	1.0 (1.00)	1.9 (1.00)	2.4 (1.00)	10.5 (.49)
Bias Test	.1 (.96)	.2 (.89)	.3 (.83)	.3 (.82)	1.8 (.14)

The maximum likelihood estimates for the imbedded EGARCH models are for the 6-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t\sqrt{h_t}, \;\; h_t \sim N(0,1).$$

Table 6
HJM Model Comparison: 12-month Rate

	Constant	Square Root	Proportional	Exponential	Combination
c_0	-28.37 (.00)	-.97 (.00)	-.82 (.00)	-.36 (.00)	-.36 (.00)
q	.02 (.00)	.08 (.00)	.10 (.00)	.12 (.00)	.12 (.00)
p	-1.00 (.00)	.93 (.00)	.94 (.00)	.97 (.00)	.97 (.00)
l	-.26 (.00)	-.34 (.01)	-.32 (.00)	-.11 (.01)	-.11 (.01)
s_0		.00 (1.00)	.00 (1.00)	.05 (.00)	.83 (.00)
s_1					11.08 (.00)
s_2					2.12 (.00)
c					-32.17 (.00)
l				-.09 (.00)	-.02 (.00)
log$L$	12218.8	12219.6	12215.1	12278.4	12284.6
Likelihood Ratio Test	131.6 >11.1	130.0 >9.49	139.0 >9.49	12.4 >7.81	
$Q(12)_e$	24.7 (.01)	29.4 (.00)	33.2 (.00)	31.6 (.00)	31.6 (.00)
$Q(12)_{e^2}$	24.6 (.01)	2.2 (.53)	6.1 (.86)	6.8 (.81)	6.8 (.81)
Bias Test	.7 (.57)	.8 (.50)	.9 (.44)	1.4 (.25)	1.4 (.25)

The maximum likelihood estimates for the imbedded EGARCH models are for the 12-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t\sqrt{h_t},\;\; h_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

Table 7
HJM Model Comparison: 24-month Rate

	Constant	Square Root	Proportional	Exponential	Combination
c_0	-14.10 (.76)	-.88 (.01)	-23.06 (.00)	-.25 (.00)	-.25 (.00)
q	.0004 (.99)	.08 (.00)	-.04 (.25)	.12 (.00)	.12 (.00)
p	.03 (.99)	.94 (.00)	-.62 (.12)	.98 (.00)	.98 (.00)
l	-2.41 (.99)	-.07 (.56)	-.21 (.76)	0.10 (.02)	0.10 (.02)
s_0		.00 (1.00)	.00 (1.00)	.01 (.00)	.08 (.00)
s_1					.07 (.00)
s_2					.07 (.00)
c					-.07 (.00)
l				-.06 (.00)	-.35 (.00)
log$L$	12423.5	12443.9	12412.4	12525.5	12528.5
Likelihood Ratio Test	210.0 > 11.1	169.2 > 9.49	232.2 > 9.49	6.0 < 7.81	
$Q(12)_e$	58.4 (.00)	57.4 (.00)	58.9 (.00)	57.4 (.00)	52.8 (.00)
$Q(12)_{e^2}$	2.7 (.00)	9.5 (.57)	41.3 (.00)	6.0 (.87)	6.1 (.87)
Bias Test	115.1 (.00)	.1 (.93)	.4 (.75)	1.1 (.34)	1.1 (.34)

The maximum likelihood estimates for the imbedded EGARCH models are for the 24-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t\sqrt{h_t},\; h_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

Table 8
HJM Model Comparison: 48-month Rate

	Constant	Square Root	Proportional	Exponential	Combination
c_0	-.27 (.00)	-.84 (.02)	-4.68 (.00)	-.42 (.15)	-.27 (.00)
q	.10 (.00)	.09 (.00)	.16 (.00)	.25 (.04)	.11 (.00)
p	.98 (.00)	.94 (.00)	.68 (.00)	1.04 (.00)	.98 (.00)
l	.08 (.18)	-.07 (.56)	-.03 (.86)	-2.10 (.16)	.08 (.18)
s_0		.00 (1.00)	-.00 (1.00)	-.00 (.00)	-.01 (.00)
s_1					.06 (.00)
s_2					.03 (.00)
c					-.10 (.00)
l				-.66 (.00)	-.06 (.00)
log$L$	12871.4	12817.4	12798.0	7825.9	12876.7
Likelihood Ratio Test	10.6 < 11.1	118.6 > 9.49	157.4 > 9.49	10101.6 > 7.81	
$Q(12)_e$	61.3 (.00)	59.9 (.00)	60.3 (.00)	32.9 (.00)	61.9 (.00)
$Q(12)_{e^2}$	11.3 (.42)	13.3 (.28)	29.7 (.00)	.6 (1.00)	11.3 (.41)
Bias Test	.3 (.85)	.9 (.45)	.4 (.73)	32.9 (.00)	.3 (.85)

The maximum likelihood estimates for the imbedded EGARCH models are for the 48-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t \sqrt{h_t}, \;\; h_t \sim N(0,1).$$

The conditional variance equations that imbed the HJM volatility models are as summarized in Panel B of Table 1.

Table 9
Forward Rate Spread Effect

Series	6-month	12-month	24-month	48-month
c_0	-.09 (.00)	-.56 (.00)	-.32 (.00)	-.28 (.00)
q	.03 (.00)	.14 (.00)	.11 (.00)	.10 (.00)
p	.99 (.00)	.96 (.00)	.98 (.00)	.98 (.00)
l	-.17 (.00)	-.07 (.56)	.08 (.10)	.07 (.29)
s_0	.002 (.00)	.006 (.00)	.003 (.00)	.001 (.00)
log$L$	12608.5	12300.4	12526.6	12871.9
$Q(12)_e$	8.2 (.69)	31.8 (.00)	58.3 (.00)	61.7 (.00)
$Q(12)_{e^2}$	1.9 (.99)	6.5 (.84)	6.3 (.85)	11.2 (.43)
Bias Test	.4 (.79)	1.8 (.14)	1.2 (.33)	31.9 (.10)

The maximum likelihood estimates for the imbedded forward rate spread EGARCH model are reported for the 6-month, 12-month, 24-month and 48-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t \sqrt{h_t}, \;\; h_t \sim N(0,1).$$

The conditional variance equation for the forward spread EGARCH model is

$$h_t = \exp(c_0 + p\log(h_{t-1}) + qg_{t-1}) + (s_0 spread_{t-1})^2,$$

where

$$g_{t-1} = \left| e_{t-1} \big/ \sqrt{h_{t-1}} \right| - \sqrt{2/p} - l e_{t-1} \big/ \sqrt{h_{t-1}}$$

$$spread_t = f_{48-month,t} - f_{6-month,t}$$

Table 10
Forward Rate Level, Maturity and Spread Effect

Series	6-month	12-month	24-month	48-month
c_0	1.61 (.00)	-.56 (.00)	-.30 (.00)	-.28 (.00)
q	.38 (.00)	.13 (.00)	.11 (.00)	.10 (.00)
p	.91 (.00)	.96 (.00)	.98 (.00)	.98 (.00)
l	-.22 (.00)	-.16 (.00)	.10 (.03)	.07 (.26)
s_0	.00 (.86)	.01 (.00)	-1.22 (.00)	.10 (.00)
s_1	3.22 (.31)	.002 (.00)	-29.29 (.00)	-.01 (.00)
s_0	-.02 (.00)	-.01 (.00)	.002 (.00)	-.001 (.01)
c	-2.72 (.21)	.02 (.00)	-102.2 (.00)	.03 (.00)
l	-.02 (.00)	-.16 (.00)	-.01 (.00)	-.04 (.00)
log$L$	12453.7	12295.1	12524.9	12871.8
$Q(12)_e$	23.7 (.01)	31.3 (.00)	57.9 (.00)	55.8 (.00)
$Q(12)_{e^2}$	3.2 (.99)	4.2 (.75)	6.0 (.87)	11.2 (.43)
Bias Test	2.2 (.09)	1.7 (.16)	1.1 (.35)	.26 (.86)

The maximum likelihood estimates for the imbedded forward rate three-factor EGARCH model are reported for the 6-month, 12-month, 24-month and 48-month forward rate series from 1990 to 1998. Numbers in the parentheses are the *p*-values. For all models, the mean equation is

$$f_t = a_0 + a_1 f_{t-1} + e_t, \quad where \;\; e_t = h_t \sqrt{h_t}, \; h_t \sim N(0,1).$$

The conditional variance equation for the forward spread EGARCH model is

$$h_t = \exp(c_0 + p\log(h_{t-1}) + q g_{t-1}) + \left[s_0 \left| f_{t-1} - c \right|^{s_1} \exp\left(l(T-t) \right) + s_0 spread_{t-1}\right]^2,$$

where

$$g_{t-1} = \left| e_{t-1} / \sqrt{h_{t-1}} \right| - \sqrt{2/p} - l e_{t-1} / \sqrt{h_{t-1}}$$

$$spread_t = f_{48-month,t} - f_{6-month,t}$$

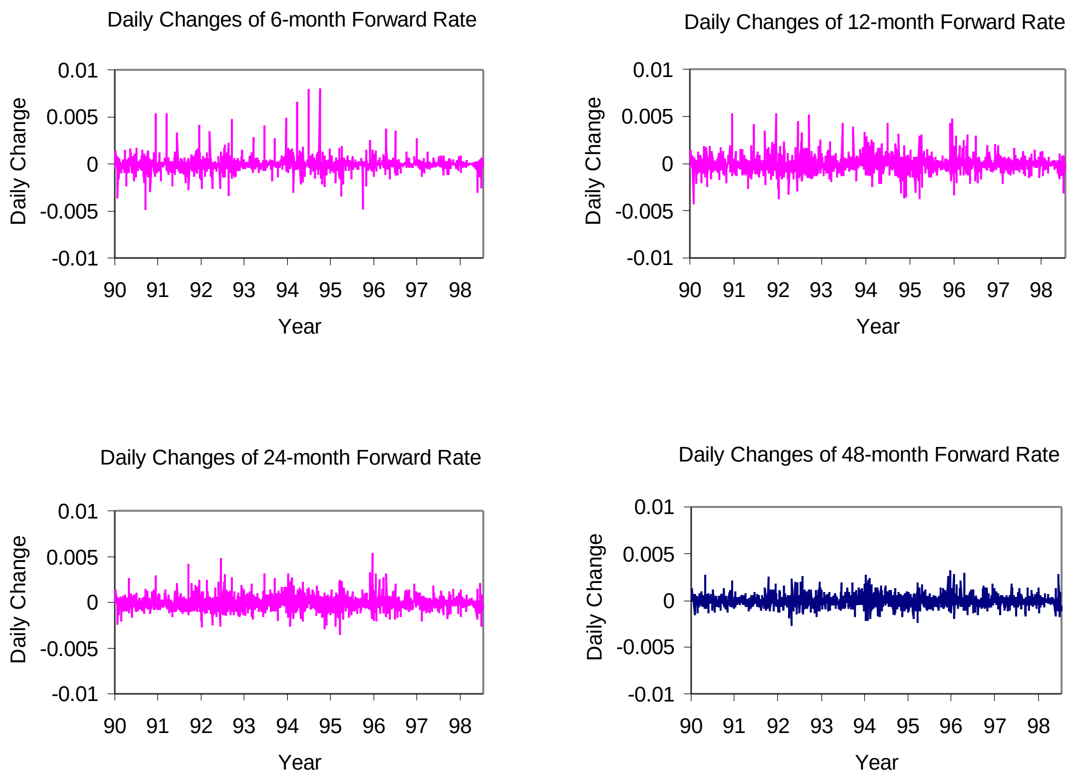

Figure 1. The daily changes of the 6-month, 12-month, 24-month and 48-month instantaneous forward rates. The instantaneous forward rates are derived from the daily Eurodollar futures prices from April 1990 to October 1998. There are 2142 observations in each series.

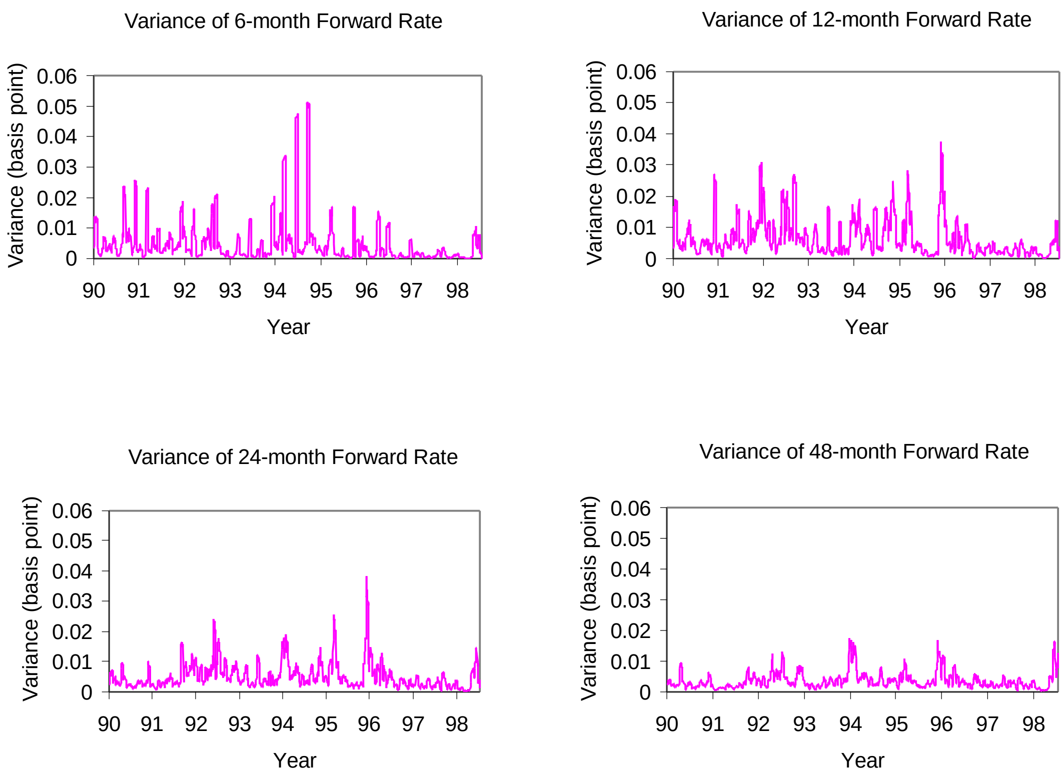

Figure 2. The 14-day moving averages of squared daily changes of the 6-month, 12-month, 24-month and 48-month instantaneous forward rates. The instantaneous forward rates are derived from the daily Eurodollar futures prices from April 1990 to October 1998. There are 2142 observations in each series.